ch - 4

Adjoint and Self - Adjoint Equations

Art. - Adjoint eqn of linear homogeneous eqn of order (n).

We define the notation $L_n(y)$ as

$$L_n(y) = a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \quad \text{--- (1)}$$

$$\text{where } L_n = a_0\frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{d}{dx} + a_n(x) \quad \text{--- (2)}$$

is called the nth order linear differential operator.

The adjoint equation of (1) is

$$\overline{L_n(y)} = (-1)^n\frac{d^n}{dx^n}[a_0(x)y] + (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}[a_1(x)y] + \cdots + (-1)\frac{d}{dx}[a_{n-1}(x)y] + a_n(x)y = 0 \quad \text{--- (3)}$$

$$\overline{L_n} = (-1)^n\frac{d^n}{dx^n}[a_0(x)] + (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}[a_1(x)] + \cdots + (-1)\frac{d}{dx}[a_{n-1}(x)] + a_n(x) \quad \text{--- (4)}$$

Note :- $(\overline{\overline{L_n}})y = L_n(y)$

$$(\overline{\overline{L_n}}) = L_n$$

Ex-1 Find the adjoint of the equation of

$$L_2(y) = a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0$$

⇒ Here with usual notation we have,

$$L_2 = a_0(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_2(x)$$

$$\bar{L}_2(y) = (-1)^2\frac{d^2}{dx^2}[a_0(x)y] + (-1)\frac{d}{dx}[a_1(x)y] + a_2(x)y$$

$$\bar{L}_2(y) = \frac{d^2}{dx^2}[a_0(x)y] - 1\cdot\frac{d}{dx}[a_1(x)y] + a_2(x)y.$$

$$= \frac{d}{dx}\left[a_0(x)\frac{dy}{dx} + a_0'(x)y\right] - \left[a_1(x)\frac{dy}{dx} + a_1'(x)y\right] + a_2(x)y$$

$$= a_0(x)\frac{d^2y}{dx^2} + a_0'(x)\frac{dy}{dx} + a_0'(x)\frac{dy}{dx} + a_0''(x)y - a_1(x)\frac{dy}{dx} - a_1'(x)y + a_2(x)y$$

$$\bar{L}_2(y) = a_0(x)\frac{d^2y}{dx^2} + [2a_0'(x) - a_1(x)]\frac{dy}{dx} + [a_0''(x) - a_1'(x) + a_2(x)]y$$

$$\bar{L}_2 = a_0(x)\frac{d^2}{dx^2} + [2a_0'(x) - a_1(x)]\frac{d}{dx} + [a_0''(x) - a_1'(x) + a_2(x)]$$

which is required adjoint equation

Ex-2 $\quad L_2(y) = x^2\frac{d^2y}{dx^2} + 7x\frac{dy}{dx} + 8y = 0$

then show that $(\overline{L_2}) = L_2$

$$\Rightarrow \quad \overline{L_2}(y) = (-1)^2\frac{d^2}{dx^2}(x^2y) + (-1)^1\frac{d}{dx}(7xy) + 8y$$

$$= \frac{d}{dx}\left[x^2\frac{dy}{dx} + 2xy\right] - \frac{d}{dx}[7xy] + 8y$$

$$= \frac{d}{dx}\left[x^2\frac{dy}{dx} + 2xy\right] - \left[7x\frac{dy}{dx} + 7y\right] + 8y$$

$$= x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + 2x\frac{dy}{dx} + 2y - 7x\frac{dy}{dx} - 7y + 8y$$

$$= x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 3y - 7x\frac{dy}{dx}$$

$$= x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 3y.$$

$$\text{Now, } (\overline{\overline{L_2}})(y) = (-1)^2\frac{d^2}{dx^2}(x^2y) + (-1)\frac{d}{dx}(-3xy) + 3y$$

$$= \frac{d}{dx}\left(x^2\frac{dy}{dx} + 2xy\right) - \left(-3x\frac{dy}{dx} - 3y\right) + 3y$$

$$= x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + 2x\frac{dy}{dx} + 2y + 3y + 3x\frac{dy}{dx} + 3y$$

$$= x^2\frac{d^2y}{dx^2} + 7x\frac{dy}{dx} + 8y = l_2(y)$$

$$(\overline{T_2}) = l_2$$